MATH 2040 Lecture 6 (Sep 26, 2016)

Recall: How to find the eigenvalues/eigenvectors of $T : V \to V$

Step 1: Fix any basis β for V, compute $A = [T]_\beta$

Step 2: Compute char. polynomial $f(t) = \det(A - tI)$

Solve $f(t) = 0 \rightsquigarrow$ get eigenvalues $\lambda_1, \dots, \lambda_k \in \mathbb{F}$

Step 3: For each λ_i, compute $N(A - \lambda_i I)$

$\rightsquigarrow$ get eigenvectors of A in $\mathbb{F}^n$

Step 4: use β to convert them back to eigenvectors of T in V

Remark: To find e.values/e.vectors of $A \in M_{n\times n}(\mathbb{F})$, one just skip Step 1 & 4.

Q: Why any basis β would work?

If $A = [T]_\beta$, $B = [T]_{\beta'}$ $\Rightarrow$ $B = Q^{-1} A Q$ similar!

FACT: (1) A, B have the same char. polynomial

$\rightsquigarrow$ same eigenvalues

(2) $\vec{v} \in \mathbb{F}^n$ e.vector of B $\Leftrightarrow$ $Q\vec{v} \in \mathbb{F}^n$ e.vector of A

Proof: Homework Exercise.

§ Diagonalizability (textbook 5.1 - 5.2)

Q: Given linear $T: V \to V$, $\exists$ basis β for V

s.t. $[T]_\beta$ = a diagonal matrix ?

(Matrix version: given $A \in M_{n\times n}(\mathbb{F})$,

$\exists$ $Q \in M_{n\times n}(\mathbb{F})$ invertible s.t.

$Q^{-1} A Q = D$ diagonal ?)

Def$^{\underline{n}}$: An eigenbasis of V for $T: V \to V$ is a ② basis of V consisting of ① eigenvectors.

Note: • $\exists$ eigenbasis $\Leftrightarrow$ T diagonalizable

• eigenbasis is NEVER unique.

Def$^{\underline{n}}$: Suppose $\lambda \in \mathbb{F}$ eigenvalue of T.

Define:

$$E_\lambda := N(T - \lambda I)$$

eigenspace of λ

$$= \left\{ \begin{matrix} \text{eigenvectors} \\ \text{of } T \text{ with} \\ \text{e.value } \lambda \end{matrix} \right\} \cup \{\vec{0}\}$$

Note: $\dim E_\lambda \geq 1$. (ie. $E_\lambda \neq \{\vec{0}\}$)

$\because$ $\exists$ e.vector $\vec{v} \neq \vec{0}$.

Example 1: (diagonalizable matrices)

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$\Rightarrow$ diagonal matrices are diagonalizable.

Example 2:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \; ; \quad f(t) = \det \begin{pmatrix} 1-t & 1 \\ 0 & 2-t \end{pmatrix}$$

$$= (1-t)(2-t) = 0$$

eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$

$\lambda_1 = 1$, $E_1 = N(A - 1 \cdot I) = N\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

$\lambda_2 = 2$, $E_2 = N(A - 2 \cdot I) = N\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$$\beta = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$ eigenbasis!

Example 3 (Non-diagonalizable)

$$\boxed{A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \; ; \quad f(t) = \det(A - tI) = \det \begin{pmatrix} 1-t & 1 \\ 0 & 1-t \end{pmatrix}$$

$$= (1-t)^2 = 0$$

$\Rightarrow$ eigenvalue: $\lambda = 1$.

$\lambda = 1$, $E_1 = N(A - 1 \cdot I) = N\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

$\dim E_1 = 1 < 2 \Rightarrow A$ is NOT diagonalizable!

$\because$ not enough eigenvectors! Pf: Exercise.

Note: All the non-diagonalizable matrices "look like this":

e.g. $\begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 2 \end{pmatrix}$ or $\begin{pmatrix} \boxed{\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}} \end{pmatrix}$ "Jordan can form"

Pf: Ex.

Example 4: (depends on $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$?)

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \text{ or } M_{2\times 2}(\mathbb{C})$$

$$f(t) = \det(A - tI) = \det \begin{pmatrix} -t & -1 \\ 1 & -t \end{pmatrix}$$

$$= t^2 + 1 = 0$$

$\Rightarrow$ No $\mathbb{R}$ eigenvalues

BuT $\mathbb{C}$ eigenvalues exist: $\lambda_1 = i$, $\lambda_2 = -i$

This matrix is NOT diagonalizable over $\mathbb{R}$

but is diagonalizable over $\mathbb{C}$.

$\because \dim E_i = \dim E_{-i} \geq 1 \Rightarrow \exists$ e.basis!

Thm: Given $T : V \to V$,

distinct e.values : $\lambda_1, \dots, \lambda_k \in \mathbb{F}$

e.spaces : $E_{\lambda_1} \cdots\cdots E_{\lambda_k}$

e.vectors : $\vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_k$

$\Longrightarrow \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ lin. indep. !

Proof: By induction k.

k = 0 : nothing to prove!

k = 1 : $\{\vec{v}_1\}$ lin. indep. $\because \vec{v}_1 \neq \vec{0}$ as e.vector

Induction argument: Assume it is true for k e.values.

Suppose $\vec{0} \neq \vec{v}_i \in E_{\lambda_i}$ where $i = 1, 2, \dots, k, k+1$.

Claim: $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}\}$ lin. indep.

Assume $\exists\, c_i \in \mathbb{F}$ s.t.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} = \vec{0} \quad \text{—} (*)$$

SCV

lin. indep.

$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$

$\Downarrow$

$c_1 = \cdots = c_n = 0$

Apply T to $(*)$

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1}) = T(\vec{0}) = \vec{0}$$

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_k\lambda_k\vec{v}_k + c_{k+1}\lambda_{k+1}\vec{v}_{k+1} = \vec{0} \quad \text{—} (\#)$$

distinct !

$(*) - \lambda_{k+1} \cdot (*)$: the $\vec{v}_{k+1}$ -term cancels.

$$\underbrace{c_1(\lambda_1 - \lambda_{k+1})\vec{v_1} + \cdots + c_k(\lambda_k - \lambda_{k+1})\vec{v}_k}_{k \text{ vectors}} = \vec{0}$$

Induction hypothesis $\Rightarrow$ $c_i(\underbrace{\lambda_i - \lambda_{k+1}}_{\neq 0}) = 0$, $i = 1, \dots, k$

$\Rightarrow$ $c_1 = \cdots = c_k = 0$.

Plug into $(*)$ $\Rightarrow$ $c_{k+1}\vec{v}_{k+1} = \vec{0}$ $\Rightarrow$ $c_{k+1} = 0$.

$\square$

<u>Corollary</u>: If $T : V \to V$, $\dim V = n$, has n distinct e.values, then T is diagonalizable. (1[st] sufficiency test)

<u>E.g.</u>: $A = \begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{pmatrix}$ $\Rightarrow$ diagonalizable.

λ_i : distinct

<u>Note</u>: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ diagonalizable

but $\exists$ 1 eigenvalue. $(1 < 2)$

Proof of 1^{st} sufficiency test: $T: V \to V$, $\dim V = n$

eigenvalues: $\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n$

eigenspaces: $E_{\lambda_1} \quad E_{\lambda_2} \quad \cdots \quad E_{\lambda_n}$

$\quad\quad\quad\quad\quad\quad \Downarrow \quad\quad \Downarrow \quad\quad\quad\quad \Downarrow$

eigenvectors: $\vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_n$

$\beta = \{ \underbrace{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n}_{\text{e.vectors}} \}$ lin. indep (by Thm)

$\Downarrow$ $\dim V = n$

e.vectors + basis = eigenbasis

$\square$

(Spectral Thm)

Hard Theorem: (2^{nd} sufficiency test)

If $A \in M_{n \times n}(\mathbb{R})$ is symmetric (i.e. $A^T = A$)

then A is diagonalizable (over $\mathbb{R}$).

Eg. $A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$ symmetric $\Rightarrow$ diagonalizable!

Note: Not "necessary":

e.g. $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ NOT symm. But diagonalizable.

$\because$ 1^{st} suff. test!

GOAL: Find $\underbrace{\text{necessary}}_{\text{mysterious now}}$ & $\underbrace{\text{sufficient}}_{\text{2 tests}}$ condition!

<u>Motivations</u>: $T : V \to V \xrightarrow{\beta} A = [T]_\beta$

$f(t) = \det(A - tI) = 0 \iff (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_k)^{m_k} = 0$

$\Rightarrow$ e.values: $\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_k$ (distinct)

e.spaces: $E_1 \quad E_2 \quad \quad E_k$

dim: $\overset{=?}{\leq} m_1 + \overset{=?}{\leq} m_2 + \cdots + \overset{=?}{\leq} m_k \leq n = \dim V$

$\because \deg f(t) = m_1 + \cdots + m_k = n$.

$= \iff$ diagonalizable.